Mathematics
Higher level
Paper 3 – calculus

Thursday 16 November 2017 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

8817 – 7208

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 5]

The function f is defined by

$$f(x) = \begin{cases} x^2 - 2, & x < 1 \\ ax + b, & x \geq 1 \end{cases}$$

where a and b are real constants.

Given that both f and its derivative are continuous at $x = 1$, find the value of a and the value of b.

2. [Maximum mark: 10]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{x}{x^2+1}y = x$ where $y = 1$ when $x = 0$.

(a) Show that $\sqrt{x^2+1}$ is an integrating factor for this differential equation. [4]

(b) Solve the differential equation giving your answer in the form $y = f(x)$. [6]

3. [Maximum mark: 12]

(a) Use the limit comparison test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ is convergent. [3]

Let $S = \sum_{n=1}^{\infty} \frac{(x-3)^n}{n^2+2}$.

(b) Find the interval of convergence for S. [9]

4. [Maximum mark: 10]

The mean value theorem states that if f is a continuous function on $[a, b]$ and differentiable on $]a, b[$ then $f'(c) = \frac{f(b) - f(a)}{b - a}$ for some $c \in]a, b[$.

The function g, defined by $g(x) = x\cos\left(\sqrt{x}\right)$, satisfies the conditions of the mean value theorem on the interval $[0, 5\pi]$.

(a) For $a = 0$ and $b = 5\pi$, use the mean value theorem to find all possible values of c for the function g. [6]

(b) Sketch the graph of $y = g(x)$ on the interval $[0, 5\pi]$ and hence illustrate the mean value theorem for the function g. [4]

5. [Maximum mark: 13]

Consider the function $f(x) = \sin(p \arcsin x)$, $-1 < x < 1$ and $p \in \mathbb{R}$.

(a) Show that $f'(0) = p$. [2]

The function f and its derivatives satisfy

$$\left(1 - x^2\right) f^{(n+2)}(x) - (2n + 1)\, x f^{(n+1)}(x) + \left(p^2 - n^2\right) f^{(n)}(x) = 0,\ n \in \mathbb{N}$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$ and $f^{(0)}(x)$ is $f(x)$.

(b) Show that $f^{(n+2)}(0) = \left(n^2 - p^2\right) f^{(n)}(0)$. [1]

(c) For $p \in \mathbb{R} \setminus \{\pm 1, \pm 3\}$, show that the Maclaurin series for $f(x)$, up to and including the x^5 term, is

$$px + \frac{p\left(1 - p^2\right)}{3!} x^3 + \frac{p\left(9 - p^2\right)\left(1 - p^2\right)}{5!} x^5 .$$ [4]

(d) Hence or otherwise, find $\lim\limits_{x \to 0} \frac{\sin(p \arcsin x)}{x}$. [2]

(e) If p is an odd integer, prove that the Maclaurin series for $f(x)$ is a polynomial of degree p. [4]